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The similarity types of primitive parallelohedra in E^d, $2 \le d \le 5$

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The similarity types of parallelohedra are defined and a complete classification of the primitive parallelohedra in E^d, $2 \le d \le 5$, is given.

1. Introduction

Following Fedorov (1885), a *parallelohedron* is a convex body, congruent copies of which admit a facet-to-facet tiling of E^3 by translations. Nowadays the term is used for arbitrary dimensions *d*. A parallelohedron is called *primitive* if, in its tiling, in each vertex, exactly $d+1$ parallelohedra meet. Only the primitive parallelohedra have generic subcones (see §4). The unique primitive type of parallelohedron in E^2 is the hexagon, and in E^3 it is the cube-octahedron (see Table 1). In an earlier paper (Engel, 2000), the contraction types of parallelohedra in E^5 were derived, and from those, for the first time, a complete enumeration of the 222 combinatorial types of primitive parallelohedra in E^5 was achieved. In this paper, a finer classification of the parallelohedra into similarity types is accomplished. It proves that the classification into contraction types, which is fundamental for the operations of contraction and extension of parallelohedra, is a special case of the more general classification into similarity types. Remarkably, for dimensions $d \le 4$, the classification into combinatorial and similarity types coincides. Beginning with dimension $d = 5$, the similarity types result in a finer classification. For dimensions $d \ge 6$ the numbers of combinatorial and similarity types formally burst (more than 200 million of primitive combinatorial types are known, but the final number will be much larger). In what follows, the definitions and notations of Engel (2000) are used.

2. Basic notations

In order to make the paper self-contained, the necessary notations are briefly stated. Let $\Lambda^d := \{\mathbf{t} \mid \mathbf{t} = m_1\mathbf{a}_1 + \ldots + m_d\mathbf{a}_d,\ m_i \in \mathbb{Z}\}$ be a *translation lattice* in Euclidean space E^d with *lattice basis* $\mathbf{a}_1, \ldots, \mathbf{a}_d$, and *Gram matrix* $\mathsf{Q} := \{q_{ij}\}$, $q_{ij} = \mathbf{a}_i\mathbf{a}_j$. We define the *Dirichlet domain* at the origin of E^d by

$$\mathsf{D}(\mathsf{Q}) := \{\mathbf{x} \in E^d \mid \mathbf{x}^t\mathsf{Q}\mathbf{x} \le (\mathbf{x}-\mathbf{t})^t\mathsf{Q}(\mathbf{x}-\mathbf{t}),\ \forall \mathbf{t} \in \Lambda^d\}. \quad (1)$$

It is a special kind of parallelohedron. Voronoï (1908*a*,*b*) conjectured that each parallelohedron is affinely equivalent to a Dirichlet domain of some translation lattice, and he proved the conjecture for primitive parallelohedra. For actual calculations we always use Dirichlet's construction for a given Gram matrix Q.

Geometrically, D(Q) is obtained as the intersection of a set of closed half-spaces $\mathsf{H}_\mathbf{t}$, each being determined through the hyperplane perpendicular to the translation vector **t** and bisecting it,

$$\mathsf{D}(\mathsf{Q}) = \bigcap_{\mathbf{t} \in \Lambda^d \setminus \{\mathbf{0}\}} \mathsf{H}_\mathbf{t}. \quad (2)$$

Only translations **t** within a ball of finite radius $2R$ contribute to D(Q), where R is the radius of the largest interstitial ball of Λ^d. Translation vectors which are facet vectors of D are denoted by **f**. For a primitive parallelohedron P, let $\mathbf{f}_1, \ldots, \mathbf{f}_d$ be the d facet vectors whose corresponding facets meet in a common vertex **v** of P. We introduce the determinant $\omega = \det(\mathbf{f}_1, \ldots, \mathbf{f}_d)$. The parallelohedron P is denoted to be *principal primitive* if for all vertices of P it holds that $\omega = 1$. For $d \le 4$, the maximal value of ω is 1, and for $d = 5$ it is 2. The k-faces of a polytope P are partially ordered with respect to inclusion. The k-faces of P, together with the empty set $\{\emptyset\}$, determine the *face lattice* $\mathcal{L}(\mathsf{P})$.

A *belt* of a parallelohedron P is a complete set of parallel $(d-2)$-faces of P. A belt contains either 4 or 6 $(d-2)$-faces. Primitive parallelohedra only contain sixfold belts.

A *zone* Z of a parallelohedron P is the set of all 1-faces (edges) E that are parallel to a zone vector $\mathbf{z}^*$, $\mathsf{Z} := \{\mathsf{E} \subset \mathsf{P} \mid \mathsf{E} \parallel \mathbf{z}^*\}$. A zone Z is called *closed* if every 2-face of P contains either two edges of Z, or else none, otherwise it is called *open*. By *zone contraction* $\mathsf{P}^\downarrow$, we understand the process of contracting every edge of a closed zone Z by the amount of its shortest edges. All zone contractions of P define the *zone-contraction lattice* $\mathcal{Z}(\mathsf{P})$. If all zones of P are open, P is denoted to be *totally contracted*. The edges of a zone Z are collected into subsets S_j^Z, $j = 1, \ldots, s(\mathsf{Z})$, according to their length l_j, $l_1 < l_2 < \ldots < l_{s(\mathsf{Z})}$. Each subset contains a multiple of $2d$ edges.

3. Classification schemes

In what follows, we give a series of successively finer classification schemes for parallelohedra:

Table 1
The combinatorial types of primitive parallelohedra in E^d, $2 \leq d \leq 5$.

d	No.	$(d-1)$-Subordination symbol	Zones	Order	Subcone	Similarity types
2	1	2_{6}	3(3)	12	3.3	
3	1	$4_{6}6_{8}$	6(6)	48	6.6	
4	1	$6_{6}8_{2}10_{12}12_{6}14_{4}$	12(9)	24	10.10	
	2	$6_{6}8_{6}12_{18}$	12(9)	72	10.10	
	3	$8_{20}14_{10}$	10(10)	240	10.10	
5	1	$8_{12}14_{2}16_{12}18_{14}20_{8}22_{4}24_{6}26_{2}28_{2}$	36(10)	2	15.15	1
	2	$8_{12}14_{2}16_{14}18_{10}20_{8}22_{10}26_{4}28_{2}$	36(10)	4	16.16	2 / 1
	3	$8_{12}14_{2}16_{16}18_{8}20_{8}22_{8}24_{4}26_{2}30_{2}$	36(11)	4	24.19	6 / 6 / 1
	4	$8_{12}14_{4}16_{8}18_{18}20_{8}24_{6}26_{4}28_{2}$	36(11)	4	23.18	4 / 3
	5	$8_{12}14_{4}16_{10}18_{12}20_{8}22_{8}24_{4}26_{2}28_{2}$	36(10)	2	18.18	4 / 3
	6	$8_{12}14_{4}16_{10}18_{12}20_{10}22_{4}24_{4}26_{6}$	36(10)	2	17.17	3 / 2
	7	$8_{12}14_{4}16_{12}18_{8}20_{10}22_{8}24_{4}26_{2}28_{2}$	36(10)	2	18.18	4 / 3
	8	$8_{12}14_{4}16_{12}18_{10}20_{8}22_{8}24_{4}28_{4}$	36(11)	8	24.20	8 / 9 / 2
	9	$8_{12}14_{4}16_{12}18_{10}20_{8}22_{10}26_{2}28_{4}$	36(11)	8	24.20	8 / 9 / 2
	10	$8_{12}14_{6}16_{4}18_{18}20_{8}22_{4}24_{4}26_{6}$	36(10)	4	19.19	6 / 6 / 1
	11	$8_{12}14_{6}16_{6}18_{14}20_{10}22_{2}24_{10}28_{2}$	36(10)	4	19.20	8 / 9 / 2
	12	$8_{12}14_{6}16_{8}18_{10}20_{12}22_{2}24_{8}26_{4}$	36(10)	4	19.19	6 / 6 / 1
	13	$8_{12}14_{6}16_{8}18_{12}20_{6}22_{10}24_{2}26_{6}$	36(10)	12	18.19	6 / 6 / 1
	14	$8_{12}14_{6}16_{10}18_{8}20_{6}22_{12}24_{6}28_{2}$	36(10)	4	19.20	8 / 9 / 2
	15	$8_{12}14_{6}18_{32}26_{12}$	36(12)	96	27.19	6 / 6 / 1
	16	$8_{12}14_{8}16_{4}18_{12}20_{10}22_{8}24_{2}26_{6}$	36(10)	4	20.22	12 / 15 / 4
	17	$8_{12}14_{8}16_{6}18_{8}20_{12}22_{6}24_{6}26_{4}$	36(10)	2	20.22	12 / 15 / 4
	18	$8_{12}14_{8}16_{10}18_{2}20_{10}22_{12}24_{4}26_{4}$	36(10)	4	20.22	12 / 15 / 4
	19	$8_{12}14_{12}16_{2}18_{6}20_{12}22_{4}24_{14}$	36(10)	12	21.26	24 / 36 / 14 / 1
	20	$8_{12}14_{12}16_{8}22_{22}24_{8}$	36(10)	48	21.26	24 / 36 / 14 / 1
	21	$8_{12}14_{12}18_{10}20_{8}22_{10}24_{8}26_{2}$	36(10)	16	21.26	24 / 36 / 14 / 1
	22	$8_{2}10_{4}12_{8}14_{16}16_{2}20_{18}22_{4}26_{4}30_{4}$	23(12)	8	15.15	2 / 1
	23	$8_{2}10_{4}12_{10}14_{12}16_{2}18_{10}20_{4}22_{10}26_{2}28_{4}30_{2}$	23(12)	4	15.15	2 / 1
	24	$8_{2}10_{4}12_{12}14_{8}16_{6}18_{4}20_{14}24_{4}28_{8}$	23(12)	8	15.15	2 / 1
	25	$8_{2}10_{4}12_{12}14_{12}16_{2}18_{4}20_{6}22_{8}24_{4}26_{4}28_{4}$	23(12)	8	15.15	2 / 1
	26	$8_{2}10_{6}12_{6}14_{12}16_{8}18_{6}20_{4}22_{4}24_{6}26_{6}30_{2}$	24(12)	2	15.15	1
	27	$8_{2}10_{6}12_{8}14_{6}16_{12}18_{8}20_{4}22_{4}26_{8}28_{4}$	24(12)	4	15.15	1
	28	$8_{2}10_{6}12_{8}14_{10}16_{6}18_{8}20_{6}24_{10}28_{6}$	24(12)	4	15.15	1
	29	$8_{2}10_{6}12_{8}14_{12}16_{6}18_{2}20_{6}22_{8}24_{4}26_{4}28_{4}$	24(12)	4	15.15	1
	30	$8_{2}10_{8}12_{4}14_{12}16_{6}18_{2}20_{10}22_{8}24_{4}26_{2}28_{4}$	24(11)	2	15.15	2 / 1
	31	$8_{2}10_{8}12_{6}14_{6}16_{10}18_{4}20_{12}24_{8}26_{2}28_{4}$	24(11)	2	15.15	2 / 1
	32	$8_{2}10_{8}12_{6}14_{8}16_{6}18_{8}20_{4}22_{8}24_{8}28_{4}$	24(11)	2	15.15	2 / 1
	33	$8_{2}10_{8}12_{6}14_{8}16_{6}18_{8}20_{6}22_{6}24_{4}26_{6}28_{2}$	24(11)	2	15.15	2 / 1
	34	$8_{2}10_{8}12_{8}14_{4}16_{6}18_{12}20_{4}22_{6}24_{4}26_{6}28_{2}$	24(11)	2	15.15	2 / 1
	35	$8_{2}10_{8}12_{8}14_{4}16_{10}18_{4}20_{8}22_{2}24_{12}26_{2}28_{2}$	24(11)	2	15.15	2 / 1
	36	$8_{2}10_{8}12_{8}14_{6}16_{6}18_{6}20_{8}22_{8}26_{8}28_{2}$	24(11)	2	15.15	2 / 1
	37	$8_{2}10_{8}12_{8}14_{8}16_{6}18_{2}20_{6}22_{10}24_{4}26_{8}$	24(11)	2	15.15	2 / 1
	38	$8_{2}10_{8}12_{10}14_{4}16_{4}18_{10}20_{4}22_{6}24_{6}26_{8}$	24(11)	2	15.15	2 / 1
	39	$8_{2}10_{8}12_{10}14_{4}16_{6}18_{6}20_{4}22_{10}24_{4}26_{8}$	24(11)	2	15.15	2 / 1
	40	$8_{2}10_{12}12_{2}14_{6}16_{6}18_{6}20_{8}22_{12}24_{2}26_{4}28_{2}$	24(10)	8	17.16	12 / 18 / 8 / 1
	41	$8_{2}10_{12}12_{2}14_{6}16_{6}18_{8}20_{6}22_{8}24_{6}26_{6}$	24(10)	2	17.16	12 / 18 / 8 / 1
	42	$8_{2}10_{12}12_{2}14_{6}16_{8}18_{6}20_{2}22_{12}24_{8}26_{4}$	24(10)	4	17.16	12 / 18 / 8 / 1
	43	$8_{2}10_{12}12_{2}14_{8}16_{4}18_{4}20_{10}22_{10}24_{4}26_{6}$	24(10)	4	17.16	12 / 18 / 8 / 1
	44	$8_{2}10_{12}12_{4}14_{6}16_{2}18_{6}20_{8}22_{12}24_{6}26_{4}$	24(10)	2	17.16	12 / 18 / 8 / 1
	45	$8_{2}10_{12}12_{4}14_{6}18_{8}20_{12}22_{8}24_{4}26_{6}$	24(10)	4	17.16	12 / 18 / 8 / 1
	46	$8_{2}10_{12}12_{6}14_{2}16_{4}18_{12}22_{10}24_{10}26_{4}$	24(10)	4	17.16	12 / 18 / 8 / 1
	47	$8_{2}10_{12}12_{6}14_{2}16_{6}18_{2}20_{16}24_{14}28_{2}$	24(10)	8	17.16	8 / 12 / 6 / 1
	48	$8_{2}10_{12}12_{6}14_{2}16_{8}18_{2}20_{10}24_{20}$	24(10)	4	17.16	12 / 18 / 8 / 1
	49	$8_{2}10_{12}12_{6}14_{4}16_{4}18_{6}22_{16}24_{10}26_{2}$	24(10)	8	17.16	12 / 18 / 8 / 1
	50	$8_{2}10_{12}12_{8}14_{2}16_{2}18_{2}20_{16}24_{18}$	24(10)	4	17.16	12 / 18 / 8 / 1
	51	$8_{2}10_{12}12_{8}14_{2}18_{10}20_{2}22_{14}24_{10}26_{2}$	24(10)	4	17.16	12 / 18 / 8 / 1
	52	$8_{2}10_{14}12_{2}14_{4}16_{4}18_{4}20_{10}22_{14}24_{4}26_{4}$	24(10)	8	17.16	12 / 18 / 8 / 1
	53	$8_{2}10_{14}12_{6}16_{4}18_{6}22_{20}24_{10}$	24(10)	24	17.16	12 / 18 / 8 / 1
	54	$8_{2}10_{14}12_{6}16_{4}20_{18}22_{2}24_{16}$	24(10)	24	17.16	12 / 18 / 8 / 1
	55	$8_{4}10_{2}12_{8}14_{8}16_{10}18_{8}20_{4}22_{10}24_{4}30_{4}$	26(12)	8	15.15	2 / 1
	56	$8_{4}10_{2}12_{10}14_{6}16_{8}18_{10}20_{2}22_{12}24_{4}28_{2}30_{2}$	26(11)	2	15.15	2 / 1
	57	$8_{4}10_{2}12_{12}14_{4}16_{8}18_{10}20_{2}22_{8}24_{8}28_{4}$	26(11)	2	15.15	2 / 1
	58	$8_{4}10_{2}12_{12}14_{4}16_{10}18_{4}20_{8}22_{8}24_{4}26_{2}28_{4}$	26(11)	4	16.16	4 / 4 / 1
	59	$8_{4}10_{2}12_{12}14_{4}16_{10}18_{4}20_{8}22_{8}24_{4}26_{4}30_{2}$	26(11)	4	16.16	4 / 4 / 1
	60	$8_{4}10_{2}12_{14}14_{2}16_{6}18_{12}20_{4}22_{6}24_{4}26_{6}28_{2}$	26(11)	2	16.16	4 / 4 / 1
	61	$8_{4}10_{2}12_{14}14_{2}16_{10}18_{4}20_{8}22_{2}24_{12}26_{2}28_{2}$	26(11)	2	16.16	4 / 4 / 1
	62	$8_{4}10_{2}12_{16}16_{8}18_{8}20_{2}22_{10}24_{4}26_{8}$	26(11)	4	16.16	4 / 4 / 1
	63	$8_{4}10_{2}12_{16}16_{8}18_{8}20_{4}22_{6}24_{6}26_{8}$	26(11)	4	16.16	4 / 4 / 1
	64	$8_{4}10_{4}12_{4}14_{4}16_{26}20_{8}26_{4}28_{8}$	27(13)	16	15.15	1
	65	$8_{4}10_{4}12_{4}14_{8}16_{12}18_{12}20_{4}22_{2}24_{4}26_{4}28_{2}30_{2}$	27(12)	4	15.15	1
	66	$8_{4}10_{4}12_{4}14_{8}16_{18}18_{4}20_{4}22_{4}26_{8}28_{4}$	27(12)	4	15.15	1

Table 1 (continued)

d	No.	($d-1$)-Subordination symbol	Zones	Order	Subcone	Similarity types
5	67	$8_{4}10_{4}12_{4}14_{10}16_{10}18_{8}20_{6}22_{4}24_{6}26_{4}30_{2}$	27(11)	2	15.15	1
	68	$8_{4}10_{4}12_{4}14_{10}16_{14}18_{6}20_{4}22_{4}26_{8}28_{4}$	27(12)	4	15.15	1
	69	$8_{4}10_{4}12_{4}14_{14}16_{4}18_{8}20_{6}22_{8}24_{4}26_{4}30_{2}$	27(11)	4	16.16	2 / 1
	70	$8_{4}10_{4}12_{4}14_{14}16_{6}18_{10}20_{4}24_{10}28_{6}$	27(12)	8	16.16	2 / 1
	71	$8_{4}10_{4}12_{4}14_{14}16_{8}18_{6}20_{4}22_{4}24_{6}26_{6}30_{2}$	27(12)	2	16.16	2 / 1
	72	$8_{4}10_{4}12_{4}14_{14}16_{10}18_{2}20_{4}22_{8}24_{4}26_{4}28_{4}$	27(12)	8	16.16	2 / 1
	73	$8_{4}10_{4}12_{4}14_{16}16_{4}18_{8}20_{6}24_{10}28_{6}$	27(12)	8	16.16	2 / 1
	74	$8_{4}10_{4}12_{4}14_{16}16_{8}20_{6}22_{8}24_{4}26_{4}28_{4}$	27(12)	8	16.16	2 / 1
	75	$8_{4}10_{4}12_{6}14_{6}16_{12}18_{10}20_{4}22_{2}24_{6}26_{6}28_{2}$	27(11)	2	15.15	1
	76	$8_{4}10_{4}12_{6}14_{8}16_{10}18_{6}20_{10}24_{8}26_{2}28_{4}$	27(11)	2	15.15	1
	77	$8_{4}10_{4}12_{6}14_{10}16_{6}18_{10}20_{4}22_{6}24_{4}26_{6}28_{2}$	27(11)	2	15.15	1
	78	$8_{4}10_{4}12_{6}14_{10}16_{8}18_{6}20_{4}22_{8}24_{6}26_{4}28_{2}$	27(11)	2	16.16	2 / 1
	79	$8_{4}10_{4}12_{6}14_{10}16_{10}20_{10}22_{8}24_{4}26_{2}28_{4}$	27(11)	2	16.16	2 / 1
	80	$8_{4}10_{4}12_{6}14_{12}16_{4}18_{8}20_{4}22_{8}24_{8}28_{4}$	27(11)	2	16.16	2 / 1
	81	$8_{4}10_{4}12_{8}14_{6}16_{8}18_{10}20_{6}22_{2}24_{4}26_{10}$	27(11)	4	16.16	2 / 1
	82	$8_{4}10_{4}12_{8}14_{6}16_{10}18_{6}20_{6}22_{6}24_{4}26_{6}28_{2}$	27(11)	2	16.16	2 / 1
	83	$8_{4}10_{4}12_{8}14_{8}16_{8}18_{4}20_{8}22_{2}24_{12}26_{2}28_{2}$	27(11)	2	16.16	2 / 1
	84	$8_{4}10_{4}12_{8}14_{8}16_{8}18_{4}20_{8}22_{8}26_{8}28_{2}$	27(11)	4	16.16	2 / 1
	85	$8_{4}10_{4}12_{8}14_{10}16_{6}18_{4}20_{4}22_{10}24_{4}26_{8}$	27(11)	2	16.16	2 / 1
	86	$8_{4}10_{4}12_{8}14_{12}18_{10}20_{4}22_{6}24_{6}26_{8}$	27(11)	4	16.16	2 / 1
	87	$8_{4}10_{6}12_{2}14_{10}16_{6}18_{8}20_{14}22_{4}24_{2}28_{6}$	27(11)	8	15.15	2 / 1
	88	$8_{4}10_{6}12_{2}14_{10}16_{8}18_{8}20_{6}22_{6}24_{8}26_{2}28_{2}$	27(10)	2	15.15	2 / 1
	89	$8_{4}10_{6}12_{4}14_{8}16_{6}18_{6}20_{12}22_{8}24_{2}26_{4}28_{2}$	27(10)	2	15.15	1
	90	$8_{4}10_{6}12_{4}14_{8}16_{8}18_{8}20_{2}22_{12}24_{6}26_{2}28_{2}$	27(10)	4	16.16	4 / 4 / 1
	91	$8_{4}10_{6}12_{4}14_{8}16_{10}18_{2}20_{10}22_{4}24_{10}26_{4}$	27(10)	4	16.16	4 / 4 / 1
	92	$8_{4}10_{6}12_{4}14_{10}16_{4}18_{4}20_{14}22_{8}24_{4}28_{4}$	27(11)	8	15.15	2 / 1
	93	$8_{4}10_{6}12_{4}14_{10}16_{4}18_{10}20_{4}22_{8}24_{6}26_{6}$	27(10)	2	15.15	2 / 1
	94	$8_{4}10_{6}12_{6}14_{6}16_{4}18_{8}20_{10}22_{10}24_{4}26_{2}28_{2}$	27(10)	2	15.15	2 / 1
	95	$8_{4}10_{6}12_{6}14_{6}16_{6}18_{6}20_{10}22_{10}24_{2}26_{4}28_{2}$	27(10)	4	15.15	2 / 1
	96	$8_{4}10_{6}12_{6}14_{6}16_{8}18_{6}20_{6}22_{8}24_{6}26_{6}$	27(10)	2	16.16	4 / 4 / 1
	97	$8_{4}10_{6}12_{6}14_{6}16_{10}18_{4}20_{2}22_{12}24_{8}26_{4}$	27(10)	2	16.16	4 / 4 / 1
	98	$8_{4}10_{6}12_{6}14_{8}16_{2}18_{6}20_{16}24_{12}28_{2}$	27(10)	4	16.16	2 / 1
	99	$8_{4}10_{6}12_{6}14_{8}16_{2}18_{10}20_{4}22_{12}24_{6}26_{4}$	27(10)	4	16.16	4 / 4 / 1
	100	$8_{4}10_{6}12_{6}14_{8}16_{4}18_{4}20_{12}22_{8}24_{4}26_{6}$	27(10)	4	15.15	2 / 1
	101	$8_{4}10_{6}12_{8}14_{4}16_{2}18_{10}20_{14}24_{12}28_{2}$	27(10)	4	16.16	4 / 4 / 1
	102	$8_{4}10_{6}12_{8}14_{4}16_{6}18_{6}20_{8}22_{10}24_{4}26_{6}$	27(10)	2	16.16	2 / 1
	103	$8_{4}10_{6}12_{8}14_{4}18_{16}20_{6}22_{8}24_{4}26_{6}$	27(10)	4	16.16	4 / 4 / 1
	104	$8_{4}10_{6}12_{8}14_{6}16_{2}18_{6}20_{10}22_{10}24_{6}26_{4}$	27(10)	2	15.15	2 / 1
	105	$8_{4}10_{6}12_{8}14_{6}16_{4}18_{4}20_{8}22_{12}24_{6}26_{4}$	27(10)	2	16.16	2 / 1
	106	$8_{4}10_{6}12_{8}14_{6}16_{4}18_{10}22_{10}24_{10}26_{4}$	27(10)	4	16.16	4 / 4 / 1
	107	$8_{4}10_{6}12_{8}14_{6}16_{6}18_{4}20_{8}24_{20}$	27(10)	4	16.16	4 / 4 / 1
	108	$8_{4}10_{6}12_{10}14_{2}16_{2}18_{10}20_{10}22_{8}24_{4}26_{6}$	27(10)	2	16.16	4 / 4 / 1
	109	$8_{4}10_{6}12_{10}14_{2}16_{4}18_{8}20_{6}22_{12}24_{6}26_{4}$	27(10)	2	16.16	4 / 4 / 1
	110	$8_{4}10_{6}12_{10}14_{2}16_{6}18_{4}20_{14}24_{14}28_{2}$	27(10)	8	16.16	2 / 1
	111	$8_{4}10_{6}12_{10}14_{4}16_{4}18_{6}20_{2}22_{14}24_{10}26_{2}$	27(10)	4	16.16	4 / 4 / 1
	112	$8_{4}10_{6}12_{10}14_{6}16_{2}20_{16}24_{18}$	27(10)	8	16.16	4 / 4 / 1
	113	$8_{4}10_{6}12_{12}14_{2}16_{2}18_{4}20_{14}24_{18}$	27(10)	4	16.16	4 / 4 / 1
	114	$8_{4}10_{6}12_{12}14_{2}18_{10}20_{4}22_{12}24_{10}26_{2}$	27(10)	4	16.16	4 / 4 / 1
	115	$8_{4}10_{6}14_{14}16_{4}18_{12}20_{6}22_{8}26_{4}28_{4}$	27(11)	8	15.15	2 / 1
	116	$8_{4}10_{8}12_{2}14_{8}16_{4}18_{6}20_{14}22_{8}24_{4}26_{2}28_{2}$	27(10)	2	15.15	2 / 1
	117	$8_{4}10_{8}12_{4}14_{6}16_{4}18_{4}20_{18}22_{2}24_{8}26_{4}$	27(10)	4	16.16	4 / 4 / 1
	118	$8_{4}10_{8}12_{4}14_{6}16_{4}18_{8}20_{6}22_{14}24_{6}28_{2}$	27(10)	4	16.16	4 / 4 / 1
	119	$8_{4}10_{8}12_{4}14_{8}16_{2}18_{6}20_{8}22_{14}24_{4}26_{4}$	27(10)	4	15.15	2 / 1
	120	$8_{4}10_{8}12_{4}14_{8}18_{8}20_{12}22_{10}24_{2}26_{6}$	27(10)	4	15.15	2 / 1
	121	$8_{4}10_{8}12_{6}14_{4}16_{4}18_{6}20_{10}22_{10}24_{6}26_{4}$	27(10)	2	16.16	4 / 4 / 1
	122	$8_{4}10_{8}12_{6}14_{4}16_{6}18_{2}20_{10}22_{14}24_{4}26_{4}$	27(10)	2	16.16	4 / 4 / 1
	123	$8_{4}10_{8}12_{8}14_{4}16_{2}18_{2}20_{16}22_{2}24_{16}$	27(10)	4	16.16	2 / 1
	124	$8_{4}10_{8}12_{8}14_{4}16_{4}18_{4}22_{20}24_{10}$	27(10)	8	16.16	4 / 4 / 1
	125	$8_{4}10_{8}12_{8}14_{4}18_{10}20_{4}22_{12}24_{10}26_{2}$	27(10)	8	16.16	2 / 1
	126	$8_{4}10_{8}14_{12}18_{12}20_{10}22_{10}26_{2}28_{4}$	27(11)	8	15.15	2 / 1
	127	$8_{6}10_{2}12_{2}14_{12}16_{4}18_{10}20_{16}22_{4}26_{2}28_{2}30_{2}$	28(11)	4	15.15	1
	128	$8_{6}10_{2}12_{4}14_{8}16_{8}18_{8}20_{14}22_{6}28_{6}$	28(11)	8	15.15	2 / 1
	129	$8_{6}10_{2}12_{4}14_{10}16_{6}18_{8}20_{10}22_{10}26_{2}28_{4}$	28(11)	8	15.15	2 / 1
	130	$8_{6}10_{2}12_{4}14_{12}16_{2}18_{6}20_{16}22_{8}24_{2}26_{2}30_{2}$	28(10)	4	16.16	2 / 1
	131	$8_{6}10_{2}12_{4}14_{12}16_{2}18_{6}20_{16}22_{8}24_{2}28_{4}$	28(10)	4	16.16	2 / 1
	132	$8_{6}10_{2}12_{6}14_{8}16_{4}18_{8}20_{14}22_{6}24_{2}26_{4}28_{2}$	28(10)	2	16.16	2 / 1
	133	$8_{6}10_{2}12_{6}14_{10}16_{2}18_{6}20_{14}22_{8}24_{4}26_{2}28_{2}$	28(10)	2	16.16	2 / 1
	134	$8_{6}10_{2}12_{6}14_{12}18_{8}20_{6}22_{16}24_{2}26_{2}28_{2}$	28(10)	4	17.19	12 / 18 / 8 / 1
	135	$8_{6}10_{2}12_{6}14_{12}18_{8}20_{6}22_{16}24_{4}30_{2}$	28(10)	12	17.19	6 / 6 / 1
	136	$8_{6}10_{2}12_{8}14_{4}16_{8}18_{6}20_{12}22_{8}24_{2}26_{4}28_{2}$	28(10)	8	16.16	4 / 4 / 1
	137	$8_{6}10_{2}12_{8}14_{6}16_{6}18_{4}20_{12}22_{10}24_{2}26_{6}$	28(10)	4	16.16	4 / 4 / 1
	138	$8_{6}10_{2}12_{8}14_{8}16_{2}18_{6}20_{16}24_{12}28_{2}$	28(10)	4	17.19	12 / 18 / 8 / 1

Table 1 (continued)

d	No.	$(d-1)$-Subordination symbol	Zones	Order	Subcone	Similarity types
5	139	$8_{6}10_{2}12_{8}14_{8}16_{2}18_{10}20_{4}22_{12}24_{6}26_{4}$	28(10)	4	17.19	6 / 6 / 1
	140	$8_{6}10_{2}12_{8}14_{8}16_{4}18_{4}20_{8}22_{14}24_{4}26_{4}$	28(10)	8	16.16	4 / 4 / 1
	141	$8_{6}10_{2}12_{8}14_{10}18_{2}20_{20}22_{2}24_{8}26_{4}$	28(10)	4	17.19	12 / 18 / 8 / 1
	142	$8_{6}10_{2}12_{8}14_{10}18_{6}20_{8}22_{14}24_{6}28_{2}$	28(10)	4	17.19	6 / 6 / 1
	143	$8_{6}10_{2}12_{10}14_{4}16_{4}18_{10}20_{6}22_{10}24_{4}26_{6}$	28(10)	4	17.19	6 / 6 / 1
	144	$8_{6}10_{2}12_{10}14_{6}16_{2}18_{6}20_{10}22_{10}24_{6}26_{4}$	28(10)	2	17.19	6 / 6 / 1
	145	$8_{6}10_{2}12_{10}14_{8}18_{4}20_{10}22_{14}24_{4}26_{4}$	28(10)	4	17.19	6 / 6 / 1
	146	$8_{6}10_{2}12_{12}14_{2}16_{6}18_{6}20_{4}22_{12}24_{10}26_{2}$	28(10)	8	17.19	6 / 6 / 1
	147	$8_{6}10_{2}12_{12}14_{4}16_{4}20_{16}22_{2}24_{16}$	28(10)	8	17.19	6 / 6 / 1
	148	$8_{6}10_{2}12_{12}14_{6}16_{2}18_{4}22_{20}24_{10}$	28(10)	24	17.19	12 / 18 / 8 / 1
	149	$8_{6}10_{2}12_{12}16_{8}18_{6}20_{12}24_{14}28_{2}$	28(10)	24	17.19	12 / 18 / 8 / 1
	150	$8_{6}10_{2}14_{12}16_{8}18_{16}20_{6}22_{6}28_{2}30_{4}$	27(12)	24	15.15	2 / 1
	151	$8_{6}10_{4}12_{2}14_{8}16_{6}18_{12}20_{8}22_{8}24_{4}26_{2}30_{2}$	30(11)	2	15.15	1
	152	$8_{6}10_{4}12_{4}14_{4}16_{6}18_{14}20_{10}22_{4}24_{6}26_{2}28_{2}$	30(10)	2	15.15	1
	153	$8_{6}10_{4}12_{4}14_{4}16_{8}18_{10}20_{10}22_{10}26_{4}28_{2}$	30(10)	4	16.16	2 / 1
	154	$8_{6}10_{4}12_{4}14_{6}16_{2}18_{18}20_{10}24_{6}26_{4}28_{2}$	30(11)	4	17.16	2 / 1
	155	$8_{6}10_{4}12_{4}14_{6}16_{6}18_{8}20_{12}22_{8}24_{4}26_{2}28_{2}$	30(10)	2	15.15	1
	156	$8_{6}10_{4}12_{4}14_{6}16_{6}18_{8}20_{12}22_{8}24_{4}26_{2}28_{2}$	30(10)	2	15.15	1
	157	$8_{6}10_{4}12_{4}14_{6}16_{6}18_{10}20_{10}22_{10}26_{2}28_{4}$	30(11)	4	17.16	2 / 1
	158	$8_{6}10_{4}12_{4}14_{6}16_{8}18_{6}20_{12}22_{8}24_{4}28_{4}$	30(11)	4	17.16	2 / 1
	159	$8_{6}10_{4}12_{4}14_{8}16_{2}18_{10}20_{14}22_{2}24_{10}28_{2}$	30(10)	4	16.16	2 / 1
	160	$8_{6}10_{4}12_{4}14_{8}16_{4}18_{8}20_{8}22_{12}24_{6}28_{2}$	30(10)	4	16.16	2 / 1
	161	$8_{6}10_{4}12_{6}14_{2}16_{6}18_{14}20_{8}22_{6}24_{4}26_{6}$	30(10)	2	17.17	3 / 2
	162	$8_{6}10_{4}12_{6}14_{2}16_{8}18_{10}20_{8}22_{10}24_{4}26_{2}28_{2}$	30(10)	2	17.17	3 / 2
	163	$8_{6}10_{4}12_{6}14_{2}16_{10}18_{4}20_{14}22_{8}24_{4}26_{2}28_{2}$	30(10)	2	17.17	3 / 2
	164	$8_{6}10_{4}12_{6}14_{4}16_{2}18_{16}20_{8}22_{6}24_{4}26_{6}$	30(10)	2	17.17	3 / 2
	165	$8_{6}10_{4}12_{6}14_{4}16_{6}18_{6}20_{16}22_{2}24_{8}26_{4}$	30(10)	2	17.17	3 / 2
	166	$8_{6}10_{4}12_{6}14_{6}16_{2}18_{10}20_{10}22_{10}24_{2}26_{6}$	30(10)	2	17.17	3 / 2
	167	$8_{6}10_{4}12_{6}14_{6}16_{4}18_{6}20_{12}22_{8}24_{6}26_{4}$	30(10)	2	17.17	3 / 2
	168	$8_{6}10_{4}12_{6}14_{6}16_{4}18_{8}20_{8}22_{10}24_{6}26_{4}$	30(10)	2	17.17	3 / 2
	169	$8_{6}10_{4}12_{6}14_{6}16_{6}18_{4}20_{8}22_{14}24_{4}26_{4}$	30(10)	2	17.17	3 / 2
	170	$8_{6}10_{4}12_{8}14_{2}16_{4}18_{12}20_{8}22_{8}24_{4}26_{6}$	30(10)	4	18.19	6 / 6 / 1
	171	$8_{6}10_{4}12_{8}14_{2}16_{6}18_{8}20_{8}22_{10}24_{6}26_{4}$	30(10)	2	18.19	6 / 6 / 1
	172	$8_{6}10_{4}12_{8}14_{2}16_{10}20_{10}22_{14}24_{4}26_{4}$	30(10)	4	18.19	6 / 6 / 1
	173	$8_{6}10_{4}12_{8}14_{6}16_{2}18_{4}20_{14}22_{2}24_{16}$	30(10)	4	18.19	6 / 6 / 1
	174	$8_{6}10_{4}12_{8}14_{6}16_{2}18_{6}20_{12}24_{18}$	30(10)	12	18.19	6 / 6 / 1
	175	$8_{6}10_{4}12_{8}14_{6}16_{6}18_{2}22_{20}24_{10}$	30(10)	12	18.19	6 / 6 / 1
	176	$8_{6}10_{4}12_{8}14_{6}18_{10}20_{6}22_{10}24_{10}26_{2}$	30(10)	4	18.19	6 / 6 / 1
	177	$8_{6}10_{4}12_{8}16_{6}18_{12}20_{12}24_{12}28_{2}$	30(10)	4	18.19	6 / 6 / 1
	178	$8_{6}10_{4}12_{8}16_{6}18_{14}20_{6}22_{8}24_{4}26_{6}$	30(10)	12	18.19	6 / 6 / 1
	179	$8_{6}10_{4}12_{8}16_{10}18_{6}20_{6}22_{14}24_{6}28_{2}$	30(10)	4	18.19	6 / 6 / 1
	180	$8_{8}10_{4}14_{2}16_{10}18_{16}20_{8}22_{4}24_{6}26_{2}28_{2}$	33(10)	2	15.15	1
	181	$8_{8}10_{4}14_{2}16_{10}18_{18}20_{8}24_{6}26_{4}28_{2}$	33(11)	4	15.15	1
	182	$8_{8}10_{4}14_{2}16_{12}18_{12}20_{8}22_{10}26_{4}28_{2}$	33(10)	4	16.16	2 / 1
	183	$8_{8}10_{4}14_{2}16_{14}18_{10}20_{8}22_{8}24_{4}26_{2}30_{2}$	33(11)	8	18.17	3 / 2
	184	$8_{8}10_{4}14_{2}16_{14}18_{10}20_{8}22_{8}24_{4}28_{4}$	33(11)	2	16.16	2 / 1
	185	$8_{8}10_{4}14_{2}16_{14}18_{10}20_{8}22_{10}26_{2}28_{4}$	33(11)	8	16.16	2 / 1
	186	$8_{8}10_{4}14_{4}16_{2}18_{32}26_{12}$	33(12)	32	19.16	2 / 1
	187	$8_{8}10_{4}14_{4}16_{6}18_{20}20_{8}24_{6}26_{4}28_{2}$	33(11)	4	19.17	3 / 2
	188	$8_{8}10_{4}14_{4}16_{8}18_{14}20_{10}22_{4}24_{4}26_{6}$	33(10)	2	17.17	3 / 2
	189	$8_{8}10_{4}14_{4}16_{10}18_{10}20_{10}22_{8}24_{4}26_{2}28_{2}$	33(10)	2	18.18	4 / 3
	190	$8_{8}10_{4}14_{4}16_{12}18_{6}20_{12}22_{8}24_{4}26_{2}28_{2}$	33(10)	2	18.18	4 / 3
	191	$8_{8}10_{4}14_{4}16_{12}18_{8}20_{10}22_{8}24_{4}28_{4}$	33(11)	8	20.19	6 / 6 / 1
	192	$8_{8}10_{4}14_{4}16_{12}18_{8}20_{10}22_{10}26_{2}28_{4}$	33(11)	8	20.19	6 / 6 / 1
	193	$8_{8}10_{4}14_{6}16_{4}18_{16}20_{10}22_{4}24_{4}26_{6}$	33(10)	4	19.19	6 / 6 / 1
	194	$8_{8}10_{4}14_{6}16_{6}18_{12}20_{12}22_{2}24_{10}28_{2}$	33(10)	4	19.20	8 / 9 / 2
	195	$8_{8}10_{4}14_{6}16_{6}18_{14}20_{6}22_{10}24_{2}26_{6}$	33(10)	12	18.19	6 / 6 / 1
	196	$8_{8}10_{4}14_{6}16_{8}18_{8}20_{14}22_{2}24_{8}26_{4}$	33(10)	4	19.19	6 / 6 / 1
	197	$8_{8}10_{4}14_{6}16_{10}18_{6}20_{8}22_{12}24_{6}28_{2}$	33(10)	4	19.20	8 / 9 / 2
	198	$8_{8}10_{4}14_{8}16_{4}18_{12}20_{8}22_{10}24_{2}26_{6}$	33(10)	4	20.22	12 / 15 / 4
	199	$8_{8}10_{4}14_{8}16_{6}18_{8}20_{10}22_{8}24_{6}26_{4}$	33(10)	2	20.22	12 / 15 / 4
	200	$8_{8}10_{4}14_{8}16_{10}18_{2}20_{8}22_{14}24_{4}26_{4}$	33(10)	4	20.22	12 / 15 / 4
	201	$8_{8}10_{4}14_{12}16_{2}18_{6}20_{12}22_{2}24_{16}$	33(10)	12	21.26	24 / 36 / 14 / 1
	202	$8_{8}10_{4}14_{12}16_{8}22_{20}24_{10}$	33(10)	48	21.26	24 / 36 / 14 / 1
	203	$8_{8}10_{4}14_{12}18_{10}20_{8}22_{8}24_{10}26_{2}$	33(10)	16	21.26	24 / 36 / 14 / 1
	204	$10_{6}12_{14}16_{18}18_{6}22_{6}24_{6}30_{6}$	19(13)	24	15.15	2 / 1
	205	$10_{8}12_{6}14_{12}16_{8}18_{10}20_{6}26_{6}28_{2}30_{4}$	20(13)	12	15.15	1
	206	$10_{8}14_{24}16_{2}18_{16}28_{8}30_{4}$	20(14)	96	15.15	1
	207	$10_{10}12_{8}14_{4}16_{8}18_{14}20_{2}22_{4}24_{4}26_{2}28_{4}30_{2}$	20(12)	8	15.15	2 / 1
	208	$10_{10}12_{8}14_{6}16_{8}18_{4}20_{14}24_{4}28_{8}$	20(12)	8	15.15	2 / 1
	209	$10_{10}12_{8}14_{10}16_{4}18_{4}20_{6}22_{8}24_{4}26_{4}28_{4}$	20(12)	8	15.15	2 / 1
	210	$10_{10}12_{10}14_{2}16_{10}18_{8}20_{6}24_{10}28_{6}$	20(12)	8	15.15	1

Table 1 (continued)

d	No.	$(d-1)$-Subordination symbol	Zones	Order	Subcone	Similarity types
5	211	$10_{14}12_6 14_2 16_6 18_8 20_8 22_8 26_8 28_2$	20(11)	4	17.16	12 / 18 / 8 / 1
	212	$10_{14}12_6 14_4 16_6 18_4 20_6 22_{10} 24_4 26_8$	20(11)	4	17.16	12 / 18 / 8 / 1
	213	$10_{14}12_6 16_6 18_{14} 20_6 22_4 24_4 26_6 30_2$	20(11)	24	17.16	8 / 12 / 6 / 1
	214	$10_{14}12_8 16_4 18_{12} 20_4 22_6 24_6 26_8$	20(11)	8	17.16	12 / 18 / 8 / 1
	215	$10_{20}12_2 16_4 20_{18} 24_{18}$	20(10)	24	20.20	120 / 240 / 150 / 30 / 1
	216	$10_{20}14_2 18_{16} 22_{10} 24_{10} 26_4$	20(10)	16	20.20	120 / 240 / 150 / 30 / 1
	217	$10_{20}14_4 18_{10} 22_{16} 24_{10} 26_2$	20(10)	16	20.20	120 / 240 / 150 / 30 / 1
	218	$10_{20}16_8 20_{18} 24_{14} 28_2$	20(10)	24	20.20	120 / 240 / 150 / 30 / 1
	219	$10_{20}16_{10} 20_{12} 24_{20}$	20(10)	20	20.20	120 / 240 / 150 / 30 / 1
	220	$10_{20}18_{20} 22_{10} 24_{10} 30_2$	20(10)	240	20.20	120 / 240 / 150 / 30 / 1
	221	$10_{22}18_{10} 22_{20} 24_{10}$	20(10)	240	20.20	120 / 240 / 150 / 30 / 1
	222	$12_{20}16_{30} 30_{12}$	15(15)	1440	15.15	1

(*a*) Two parallelohedra P and P' are said to be combinatorially equivalent, $\mathsf{P}' \overset{\text{comb}}{\simeq} \mathsf{P}$, and belong to the same *combinatorial type*, if there exists a combinatorial isomorphism $\tau : \mathcal{L}(\mathsf{P}) \to \mathcal{L}(\mathsf{P}')$.

(*b*) They are denoted to be contraction equivalent, $\mathsf{P}' \overset{\text{contr}}{\simeq} \mathsf{P}$, and belong to the same *contraction type*, if

(i) there exists a lattice isomorphism $\kappa : \mathcal{Z}(\mathsf{P}) \to \mathcal{Z}(\mathsf{P}')$;

(ii) there exists a combinatorial isomorphism for each $\widetilde{\mathsf{P}} \in \mathcal{Z}(\mathsf{P})$, $\kappa : \widetilde{\mathsf{P}} \mapsto \widetilde{\mathsf{P}}' \overset{\text{comb}}{\simeq} \widetilde{\mathsf{P}}$.

(*c*) They are denoted to be similarly equivalent, $\mathsf{P}' \overset{\text{sim}}{\simeq} \mathsf{P}$, and belong to the same *similarity type*, if

(i) there exists a combinatorial isomorphism $\tau : \mathcal{L}(\mathsf{P}) \to \mathcal{L}(\mathsf{P}')$;

(ii) for each zone Z_i, they have the same order of subsets $\mathsf{S}_j^{\mathsf{Z}_i}$, $j = 1, \ldots, s(\mathsf{Z}_i)$.

(*d*) They are denoted to be affinely equivalent, $\mathsf{P}' \overset{\text{aff}}{\simeq} \mathsf{P}$, and belong to the same *affine type*, if there exists an affine mapping $\mathsf{A} : \mathsf{P}' = \mathsf{AP}$, where A is any $d \times d$ invertible real matrix.

In what follows we propose various schemes to characterize the equivalence type of a parallelohedron. Note that for each $\mathsf{D}(\mathsf{Q}')$, $\mathsf{Q}' \in \Phi^+$ (for the definition of Φ see §4), using the same algorithm for its calculation, an identical face lattice $\mathcal{L}(\mathsf{D})$ is obtained. This allows us to construct *relative equivalence schemes*. By writing down for each facet of P the numbers of all subordinated vertices in increasing order, we obtain the *relative polytope scheme* which uniquely characterizes the combinatorial type within Φ^+.

A rough identification of the combinatorial type is obtained by the subordination symbol. For any k, $1 < k < d$, let $n_i^{(k)}$ be the number of k-faces of P which have subordinated $f_i^{(k)}$ $(k-1)$-faces, $i = 1, \ldots, r$. The *k-subordination symbol* is defined by

$$f_1^{(k)}{}_{n_1^{(k)}} f_2^{(k)}{}_{n_2^{(k)}} \ldots f_r^{(k)}{}_{n_r^{(k)}}, \tag{3}$$

with $f_1^{(k)} < f_2^{(k)} < \ldots < f_r^{(k)}$. For primitive parallelohedra in E^d, $d \le 5$, the $(d-1)$-subordination symbol was found to be sufficient for a unique identification. A complete identification is obtained by the *unified polytope scheme* described in Engel (1991) which, however, is very time-consuming to determine.

An identification of the similarity type is obtained by the similarity scheme. A zone Z is called *admissible*, if $s(\mathsf{Z}) > 1$. By comparing in a predetermined order for each admissible zone Z_i the lengths $l_h^{\mathsf{Z}_i}$ and $l_k^{\mathsf{Z}_i}$, $1 \le h < k \le s(\mathsf{Z}_i)$, we obtain a sequence of comparison operators '$<, =, >$' which defines the *relative similarity scheme*. Note that for *zonohedral parallelohedra*, where in each zone Z_i the edges are of the same length $l_1^{\mathsf{Z}_i}$, the similarity scheme is empty. Under any affine transformation A, the lengths $l_j^{\mathsf{Z}_i}$, $j = 1, \ldots, s(\mathsf{Z}_i)$, keep a fixed proportion λ for each zone $\mathsf{Z}_i \subset \mathsf{P}$, $i = 1, \ldots, N_z$,

$$\lambda^{\mathsf{Z}_i}(\mathsf{A})\{l_1^{\mathsf{Z}_i} : l_2^{\mathsf{Z}_i} : \ldots : l_{s(\mathsf{Z}_i)}^{\mathsf{Z}_i}\}. \tag{4}$$

Thus the similarity scheme is invariant under affine transformations, and non-equivalent similarity types correspond to different affine types.

4. Subdivision of the cone $\mathcal{C}$

We partition the *open cone of positive definite quadratic forms*

$$\mathcal{C}^+ = \{\mathsf{Q}' \mid \mathbf{x}^t\mathsf{Q}'\mathbf{x} > 0\} \tag{5}$$

into connected open *subcones of equivalent combinatorial types of parallelohedra*,

$$\Phi^+(\mathsf{P}) = \{\mathsf{Q}' \in \mathcal{C}^+ \mid \mathsf{P}(\mathsf{Q}') \overset{\text{comb}}{\simeq} \mathsf{P}\}. \tag{6}$$

By Φ we denote the closure of Φ^+. A limiting surface of a combinatorial subcone $\Phi(\mathsf{P})$ is given by a *coincidence condition*, that is, at least $d+1$ facets meet in a common vertex $\mathbf{v} \subset \mathsf{P}$. Let $\mathbf{f}_1, \ldots, \mathbf{f}_{d+1}$ be the corresponding facet vectors. Since $\mathbf{f}_1, \ldots, \mathbf{f}_d$ form a basis of a sublattice of Λ^d of index ω, it follows that $\mathbf{f}_{d+1} = \alpha_1\mathbf{f}_1 + \ldots + \alpha_d\mathbf{f}_d$, $\alpha_i \in \mathbb{Z}/\omega$. Thus, the surface is determined by (see Engel, 1998)

$$\sum_{i=1}^{d} \alpha_i(\alpha_i - 1)\mathbf{f}_i^{\,t}\mathsf{Q}\mathbf{f}_i + 2\sum_{i=1}^{d-1}\sum_{j=i+1}^{d} \alpha_i\alpha_j\mathbf{f}_i^{\,t}\mathsf{Q}\mathbf{f}_j = 0. \tag{7}$$

As a shorthand we write $\mathbf{nq} = 0$, where $\mathbf{n}^t = (h_{11}, h_{12}, \ldots, h_{dd})$ is the *wall normal*, and $\mathbf{q}^t = (q_{11}, q_{12}, \ldots, q_{dd})$ represents a vector in $\mathcal{C}^+$. It is linear in the q_{ij}, and thus Φ is a polyhedral subcone.

If for some $\mathsf{Q}' \in \Phi$, and for a certain zone Z, with subsets $\mathsf{S}_h^{\mathsf{Z}}$ and $\mathsf{S}_k^{\mathsf{Z}}$, respectively, either the length $l_h = 0$, or else $l_h = l_k$, $1 \le h < k \le s(\mathsf{Z})$ then Q' lies on a limiting wall. The coincidence condition $l_h = 0$ determines under contraction of any edge of $\mathsf{S}_h^{\mathsf{Z}}$ a limiting wall of combinatorial type $\mathbf{nq} = 0$ of Φ. The coincidence condition $l_h = l_k$ determines under contraction of any edge of S_h^Z and S_k^Z, two limiting walls $\mathbf{n}_1\mathbf{q} = 0$ and $\mathbf{n}_2\mathbf{q} = 0$, respectively, which do not necessarily belong to Φ. Then the *simultaneity condition* $\mathbf{n}_1\mathbf{q} = \mathbf{n}_2\mathbf{q}$ determines a limiting *inner wall* of similarity type which intersects Φ. An inner wall is of contraction type, exactly if Z is closed, and the walls $\mathbf{n}_1\mathbf{q} = 0$ and $\mathbf{n}_2\mathbf{q} = 0$, respectively, are boundary walls of Φ. Thus contraction equivalence is a special case of the more general similarity equivalence.

We now describe an algorithm to determine the walls of a subcone $\Phi(\mathsf{Q}_0)$. Let $\mathbf{q}_0$ be the corresponding vector in $\mathcal{C}^+$, and choose $\mathbf{u} \in \mathcal{C}^+$ such that $\mathbf{q}_0\mathbf{u} = 0$.

(1) Starting at $\mathsf{Q}_0 \in \Phi^+$, move Q' along the straight line $\mathbf{q}_0 \pm \lambda\mathbf{u}$, $0 < \lambda \in \mathbb{R}$, until it comes within a small ε-distance, $0 < \varepsilon < \delta \in \mathbb{R}$, to the nearest wall which intersects this line. A coincidence condition occurs when Q' hits the wall. By this, both vertices subordinated to a certain edge coincide in the vertex $\mathbf{v}$. The facets meeting in $\mathbf{v}$ are determined, and the wall normal $\mathbf{n}$ is evaluated.

(2) Suppose p wall normals are determined. Choose $\mathbf{u}$ perpendicular to $\mathbf{n}_1, \ldots, \mathbf{n}_p$, and repeat (1) until $p = \binom{d+1}{2}$, and then calculate the simplicial cone $\mathsf{K} \supseteq \Phi$.

(3) Calculate the edges of K. Since the wall normals $\mathbf{n}_i$ have integral representations, the edge vectors $\mathbf{w}_j^*$ also have integral representations.

(4) For each edge $\lambda\mathbf{w}_j^* \notin \Phi$, a straight line is drawn from Q_0 to $\mathbf{w}_j^*$, and the wall intersecting that line closest to Q_0 is determined. If new walls of Φ are obtained, return to (3). Otherwise if all edges of K lie on the boundary of Φ then $\mathsf{K} = \Phi$ is complete.

Let $\lambda\mathbf{w}_i^*$, $i = 1, \ldots, r$, be the edges of the subcone $\Phi(\mathsf{P})$. An edge vector $\mathbf{w}_i^*$ is either:

(i) a *ray vector* lying on the boundary of $\mathcal{C}$, and thus has a representation as a tensor product $\mathbf{z}^* \otimes \mathbf{z}^*$, where $\mathbf{z}^*$ is a zone vector of a closed zone of P. The tensor product $\mathbf{z}^* \otimes \mathbf{z}^*$ has zero determinant;

(ii) a *generic inner edge form* of $\mathcal{C}^+$ having positive determinant; or

(iii) a *non-generic inner edge form* of $\mathcal{C}^+$ having zero determinant, that is, it is a generic inner edge form of a cone $\overline{\mathcal{C}}^+$ of lower dimension $\binom{k+1}{2}$, $k < d$.

Inner edge forms cannot be written as tensor products. They correspond to totally contracted parallelohedra. Inner edge forms $\mathbf{w}^*$ occur only in dimensions $d \geq 4$.

In order to partition $\Phi(\mathsf{P})$ into connected open *subcones of equivalent similarity types of parallelohedra*,

$$\phi^+(\mathsf{P}) = \{\mathsf{Q}' \in \Phi^+(\mathsf{P}) \mid \mathsf{P}(\mathsf{Q}') \overset{\text{sim}}{\simeq} \mathsf{P}\}, \qquad (8)$$

an analogous algorithm is used as described above, with coincidence condition $l_h = l_k$ for some zone Z_i. The two limiting walls $\mathbf{n}_1\mathbf{q} = 0$ and $\mathbf{n}_2\mathbf{q} = 0$, respectively, are determined, and finally the simultaneity condition $\mathbf{n}_1\mathbf{q} = \mathbf{n}_2\mathbf{q}$ gives the required limiting internal wall of similarity type which intersects Φ. Besides the generic subcones ϕ_i^+, we also have to consider the inner k-faces of $\phi_i \in \Phi^+$. They are subcones $\overline{\phi}_j$ of lower dimension which correspond to special similarity types having '=' operators in their relative similarity scheme. Let $\overline{\mathsf{W}}_{h_1}, \ldots, \overline{\mathsf{W}}_{h_r}$ be the inner walls of a subcone ϕ_h. Then any subset $\mathsf{H} = \{h_{i_1}, \ldots, h_{i_s}\} \in \{h_1, \ldots, h_r\}$, $1 \leq s \leq r$, gives a subcone

$$\overline{\phi}_\mathsf{H} = \bigcap_{i \in \mathsf{H}} \overline{\mathsf{W}}_i. \qquad (9)$$

Let $\lambda_i\mathbf{w}_i^*$, $\lambda_i > 0$, $i = 1, \ldots, r$ be the edge forms of a subcone of a given equivalence type P of parallelohedra. Then any set of $\lambda_i > 0$, $i = 1, \ldots, r$ gives a Gram matrix interior to the subcone by

$$\mathsf{Q} = \sum_{i=1}^{r} \lambda_i\mathbf{w}_i^*. \qquad (10)$$

Table 2
The numbers of k-faces of primitive parallelohedra in E^d, $2 \leq d \leq 5$.

d	$\omega_{\max}$	N_0 ($\omega = 2$)	N_0	N_1	N_2	N_3	N_4	Belts
2	1		6	6				6_1
3	1		24	36	14			6_6
4	1		120	240	150	30		6_{25}
5	2	12	708	1770	1536	534	62	6_{89}
	1		720	1800	1560	540	62	6_{90}

5. Results

Starting from an arbitrary parallelohedron P_0 in E^d, $2 \leq d \leq 5$, we calculated its subcone $\Phi_0(\mathsf{P}_0)$. Determining for each wall of Φ_0 all its neighbouring subcones, and classifying them according to their combinatorial type, we get new types. Continuing in this way, we finally obtained all combinatorial types of primitive parallelohedra in E^d. In Table 2, general properties of primitive parallelohedra in E^d are stated. The first column states the dimension d. The second column states the maximal value of ω, and the third column gives the number of vertices which have $\omega = 2$. The further columns give the numbers N_k of k-faces. The last column gives the number of sixfold belts.

For each combinatorial type of parallelohedra, we have subdivided its subcone Φ_i into subcones ϕ_{ij} of similarity types. Finally the special subcones $\overline{\phi}_{ijh}$ of lower dimension were determined. For dimensions $d \leq 4$ the subcones ϕ_i coincide with the subcones Φ_i. For dimension $d = 5$ it was found that all subcones ϕ_{ij} are simplicial cones having $\binom{d+1}{2}$ walls each. This is no longer true for dimensions $d \geq 6$. In Table 1 are listed specifications of the combinatorial types of primitive parallelohedra in E^d, $2 \leq d \leq 5$, together with the numbers of similarity types. In the list for $d = 5$, the types Nos. 1 to 21 correspond to primitive parallelohedra of the first kind with 708 vertices. Nos. 22 to 222 correspond to principal primitive parallelohedra with 720 vertices. Parallelohedra in dimension $d = 2, 3$ are zonohedral. In each dimension there exists exactly one type of primitive parallelohedron which is zonohedral. For $d = 4$, it is the type No. 3, and for $d = 5$, it is the type No. 222. Zonohedral parallelohedra are described in Engel (2005). In Table 1, the column 'zones' gives the numbers of zones N_z, whereof the number of closed zones is given in brackets. The column 'order' gives the order h_{aut} of the group of automorphisms of the face lattice $\mathcal{L}(\mathsf{P})$, obtained by the unified polytope scheme (Engel, 1991). For the order $h_{\rm sym}$ of the symmetry group (group of isomorphisms) of P it holds that $h_{\rm sym} \le h_{\rm aut}$. In the column 'subcone', the numbers of walls $N_{\rm w}$ and of extreme edges $N_{\rm e}$ of the subcone Φ_i are given by $N_{\rm w}.N_{\rm e}$. In the last column are stated the numbers $N_n/N_{n-1}/\ldots/N_{n-r}$ of similarity types with subcones of dimension $n = \binom{d+1}{2}, n-1, \ldots, n-r$, where r depends on the maximal number of inner walls of the $\phi_i \subset \Phi(\mathsf{P})$. In dimension $d = 5$, r is at most 4, but for $d > 5$, r reaches $\binom{d+1}{2} - 1$. For each subcone Φ_i in E^5, $i = 001, \ldots, 222$, the wall normals and the edge forms are given in records WNi and EFi, respectively. For the subcones of similarity types ϕ_{ij}, $j = 001, \ldots$, the relevant data are stored in records SWNij and SEFij, respectively. The records may be obtained upon request directly from the author, or from the supplementary material.[1]

As a final result, we obtained: *In E^5, there exist 6442 similarity types of primitive parallelohedra, whereof 1906 are generic, and 2870, 1419, 240 and 7 are special having subcones of dimension 14, 13, 12 and 11, respectively. All subcones are simplicial*.

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[1] Supplementary material for this paper is available from the IUCr electronic archives (Reference: WX5003). Services for accessing these data are described at the back of the journal.